Optimal allocation of the diversification capital

El Gharib Meryem Guenneugues Alexandre Leroy Antoine
Levavasseur Grégoire

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Abstract

The aim of our “Bureau d’études”, hereinafter “the project”, is to determine, within the frame of Solvency II, the optimal method(s) to allocate the diversification advantage observed in financial portfolios. These methods are to be implemented in R or VBA.

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Chapter 1

The frame

1.1 Goals of this project

First, here is an example to introduce the aims of this project. Let us consider two entities A and B. A has a risk of 10 (the concept of risk will be explained after) and B has a risk of 5. But, with the diversification effect, the coalition A+B has only a risk of 12. If 10 and 5 represent money that A and B have put aside, how could the benefice of 3 (due to the diversification) be allocated? What is the optimal allocation?

We will try to give an answer to these questions in this paper. In order to do so, we will first introduce the different allocation methods found in the various articles provided by William Gehin[1], the actuary in charge of this project, and in other relevant articles. Then we will compare these methods. After deciding which ones will be relevant for the optimal allocation, we will implement them in VBA or R. The time of execution of the algorithms shall be capped. Eventually, we will run the programs with actual data provided by William Gehin. If the results are satisfactory, the program could be used by BNP-Paribas Cardif.

1.2 Notations

We will give here some general definitions used in this document.

A **business unit** (hereinafter b.u.) refers to a daughter company, which means a logical element or segment of a company representing a specific business function. In our case, it's a portfolio, which means a part of the whole company. Here we have to allocate to each b.u. a part of the diversification profit. Sometimes, the term "module" will be used instead of b.u.: the meaning remains the same.

In economic and financial worlds, the **capital** is a set of goods and wealth which is likely to bring more revenue.

We usually name SCR the global **Solvency capital requirement** under Solvency 2, and $SCRi$ the solvency capital of each b.u.

1.3 About balance sheet

1.3.1 The own funds as fundamental stone of a balance sheet

Generally, we can define the balance sheet of a firm in three parts as following: the own funds (also named owner's equity) enabling to start up the business, that's the fundamental stone of liabilities (and of the balance sheet), the rest of liabilities and the assets [Figure 1.3.1].

A fundamental accounting rule is that the assets must equal the liabilities. Generally we can say that the goal of a firm is to increase its assets, increasing also the liabilities. This augmentation makes profit for the company.

[1] Actuary in BNP-Paribas Cardif

Figure 1.3.1: Simplified balance sheet

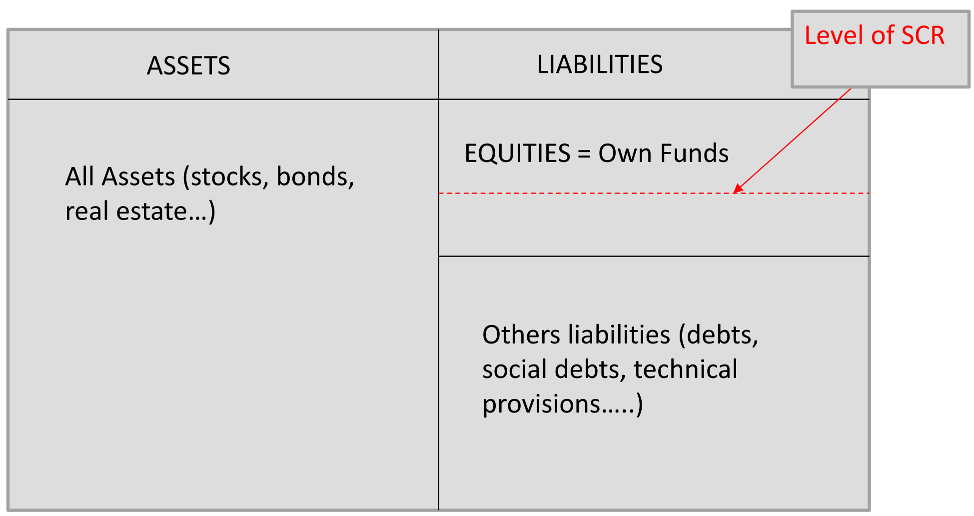
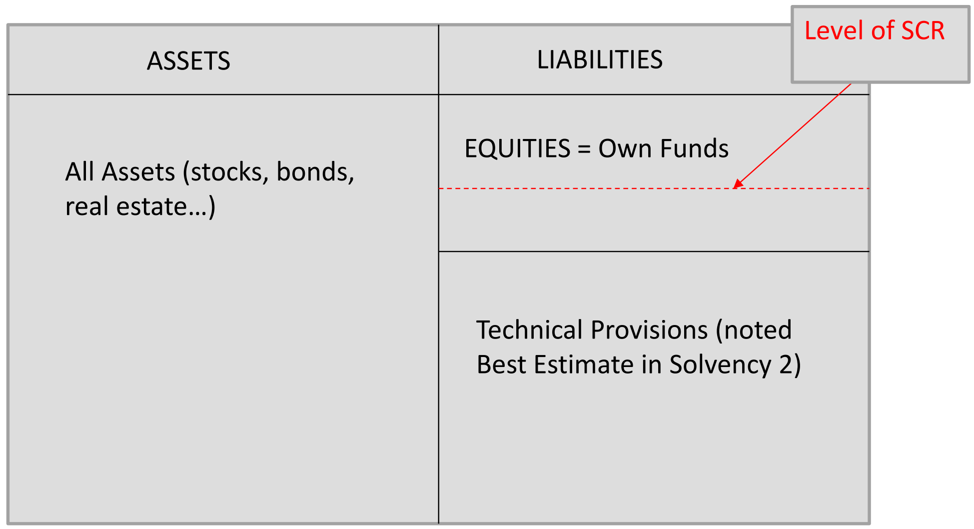


Figure 1.3.2: Simplified balance sheet of an insurance company



1.3.2 Balance of an insurance company

Insurance industry is very particular due to its reversal of production cycle: it sells a product which price will be known only in the future. That is why the insurance company must establish sufficient technical provisions to ensure its financial soundness, and not too high to make profits. So the simplified balance sheet of an insurance company reflects this particularity [Figure 1.3.2].

1.3.3 Our model of computation

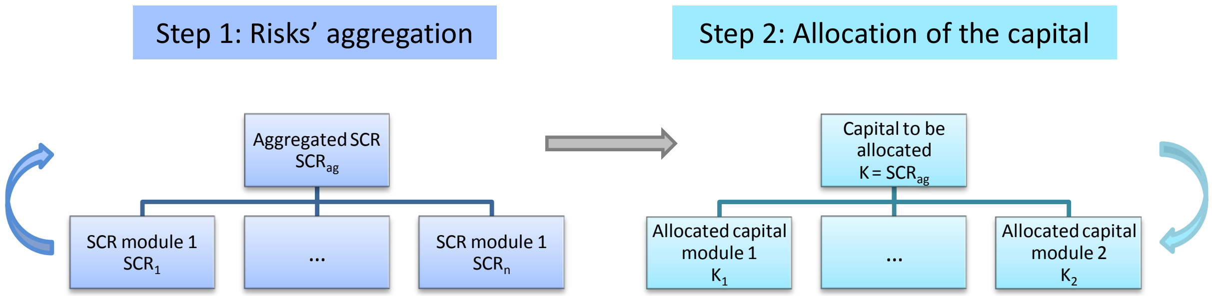
Following the above model, in our framework we make hypothesis of constant technical provisions. That implies that assets losses equal own funds losses. This is a very strong hypothesis because in fact technical provisions are not always constant when the assets fluctuate: especially in the case of rate fluctuations, assets and liabilities fluctuate (due to discount factors).

So here simulating asset losses is simulating own fund losses. This allows to quantify the level of SCR (see §1.8 Solvency 2). In practice, insurance companies have more owner's equity than SCR, especially in non-life companies. Indeed, non-life insurance activities have to force important fluctuations. Besides, they cover shorter periods (typically one year) than life insurance companies (see for instance [8]).

1.4 Capital allocation

The capital allocation is a procedure which decides how businesses and companies divide their financial resources of capital to several projects and processes. It is an important way for firms and especially insurance companies to manage risk. Indeed we can prove that the sum of the risk of different firm's portfolios is larger than the risk of the whole firm [2]. This diversification effect is seen during a previous step: the risk aggregation [Figure 1.4.1]. First of all, the allocation of capital requires the establishing of a pair made up of a risk measure and an allocation method, such as (VAR, Shapley) or (TVAR, Euler).

Figure 1.4.1: Link between risk aggregation and capital allocation



1.5 Definitions

1.5.1 Risk measure

A risk measure ρ quantifies the level of risk of a random variable X, that could be a firm's net worth at a specified point of the future or a variation of equities between today and a future date. For this report, let us consider X the SCR at the current date t:

$$\boxed{X = SCR(t).}$$

So, $\rho(X)$ is a real number which determines the risk capital needed by the company to ensure that the SCR of the firm will be acceptable - from the standpoint of a controller for instance. The choice of the risk measure is important. Indeed, two different risk measures may provide two totally different interpretations. Besides, it is common to expect a risk measure to satisfy some properties, defined below. In this case, the risk measure is considered as **coherent**.

1.5.2 Coherent risk measure

Let us consider two bounded variables X and Y that represent a risk variable. For instance, we suppose here that X and Y represent the SCR at the same moment t of two different b.u. A risk measure is coherent if it satisfies the following properties:

Sub-additivity: For all bounded random variables X and Y,

$$\rho(X+Y) \leq \rho(X) + \rho(Y).$$

This property guarantees that the risk of the holding is smaller (or equal in the case there is no diversification) than the sum of the risk of each b.u. In other words, gathering two entities does not create extra risk.

Monotonicity: For all bounded random variables X, Y such that X$\leq$ Y,

$$\rho(X) \leq \rho(Y).$$

This property implies, that if the loss X is always less than the loss Y, then X shall always be less risky than Y.

Positive homogeneity: For all $\lambda \geq 0$ and bounded random variables X,

$$\rho(\lambda X) = \lambda \rho(X).$$

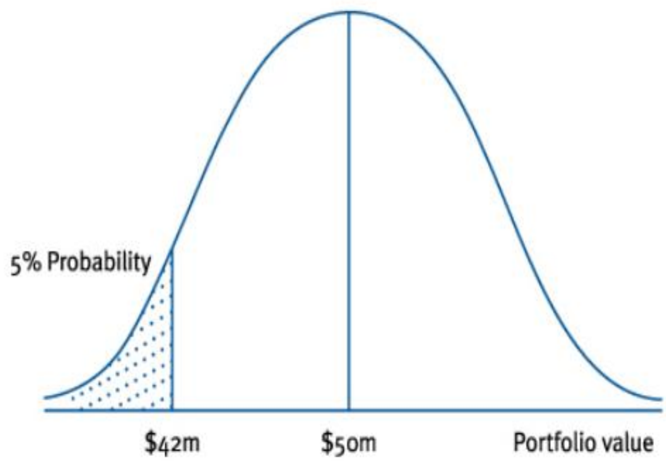
This property says the risk of b.u. multiplied by a number is equal to this number multiplied by the risk of the b.u. This case is encountered with no diversification effect.

Translation invariance: For all $\alpha \in \mathbb{R}$ and bounded random variable X,

$$\rho(X + \alpha \tau_f) = \rho(X) + \alpha.$$

Here τ_f is the riskless rate, which means a riskless investment doesn't provide diversification.

Figure 1.6.1: An example of VaR at a 5 % confidence level



1.6 Value-at-risk - VaR

Value-at-Risk is a widely used risk measure. For instance, under Solvency 2, the SCR is based on a Value-at-Risk measure calibrated to a 99.5% confidence level over a 1-year time horizon. Here is an example given by the European Comission: if VaR is measured over a one-year period at a confidence level of 99.5% then this corresponds to the worst loss one would expect to occur in a single year over the next two hundred years. The mathematical definition is:

$$VaR_X(p) = F_X^{-1}(p) = inf\{x \in \mathbb{R} | F(x)_X \geq p\}, \quad p \in [0, 1]$$

with F_X the distribution function of the random variable X.

Proposition: VaR satisfies the properties of monotonicity, positive homogeneity and translation invariance. Unfortunately, the sub-additivity is not satisfied by VaR. Thus, it is impossible to be sure to obtain a diversification effect by using VaR. In practice, and it will be shown later in this paper, it is usually sub-additive under some conditions. For theses affirmations, see for instance [9].

Let us give a simple example for a standard normal distribution: At a level of 99.5 %, 99.5 % of the realisations of the random variable are under 2.576. That implies

$$VaR_X(0.995) = 2.576.$$

Let us recall that a Gaussian distribution is defined over the real line, so it means that the realisations are between $-\infty$ and 2.576. For a random variable Y normally distributed with mean μ and standard deviation σ, the result is, using the above properties of positive homogeneity and translation invariance:

$$VaR_Y(0.995) = \mu + 2.576\sigma.$$

1.7 Tail Value-at-risk - TVaR / Conditional Tail Expectation - CTE

TVaR and CTE are actually the same. The term TVaR will be used in the document. By its definition, it is possible to know with VaR how often the worst cases will occur at some confidence level. The problem is that it gives no information about theses worst cases. TVaR tries to provide more information by determining how much, on average, the worst cases will cost to the firm. Thus the name of "tail", because TVaR gives information about the tail of the distribution. One mathematical definition is:

$$TVaR_X(p) = E[X|X > VaR_X(p)].$$

One can read this mathematical definition "at a level confidence p, TVaR of X is the expected loss knowing that X is greater that $VaR_X(p)$".

Proposition: The Tail value at risk is a coherent risk measure [10].

Let us give a simple example for a standard normal distribution: at a level of 99 %, the average of values over 99 % of the realisations of the random variable is 2.66. We notice that is not so far from VaR of level 99.5%.

We have estimated this value with a Monte-Carlo simulation, because quantiles of Gaussian distributions are not easy to compute (VaR of level 99 % is 2.3263).

As far as that goes, for a random variable Y normally distributed of mean μ and standard deviation σ, using the above properties of positive homogeneity and translation invariance, the result is:

$$TVaR_Y(p) = \mu + 2.66\sigma.$$

1.8 Solvency 2

Solvency 2 is a new European regulation which uniforms the restrictions of risk management for insurance companies. It should become effective the 1st of January, 2016. Its main goal is to ensure the financial soundness of insurance undertakings. In particular, insurers must have sufficient available resources to cover both a Minimum capital requirement (MCR)and a Solvency capital requirement (SCR).

In case of diversification, this SCR has to be distributed. The aggregated SCR is calculated with the standard formula provided by Solvency 2, given hereinafter. An intern model, justified and approved, may be used, and consists in computing a level of own funds enabling the financial soundness in 99.5% of cases. In the present case, BNP Paribas Cardif has decided to apply the standard formula. This formula consists in applying predefined shocks to each asset classes, without including own risks of each firm (1.8.1). The correlation matrix is provided by EIOPA (European Insurance and Occupational Pensions Authority) too.

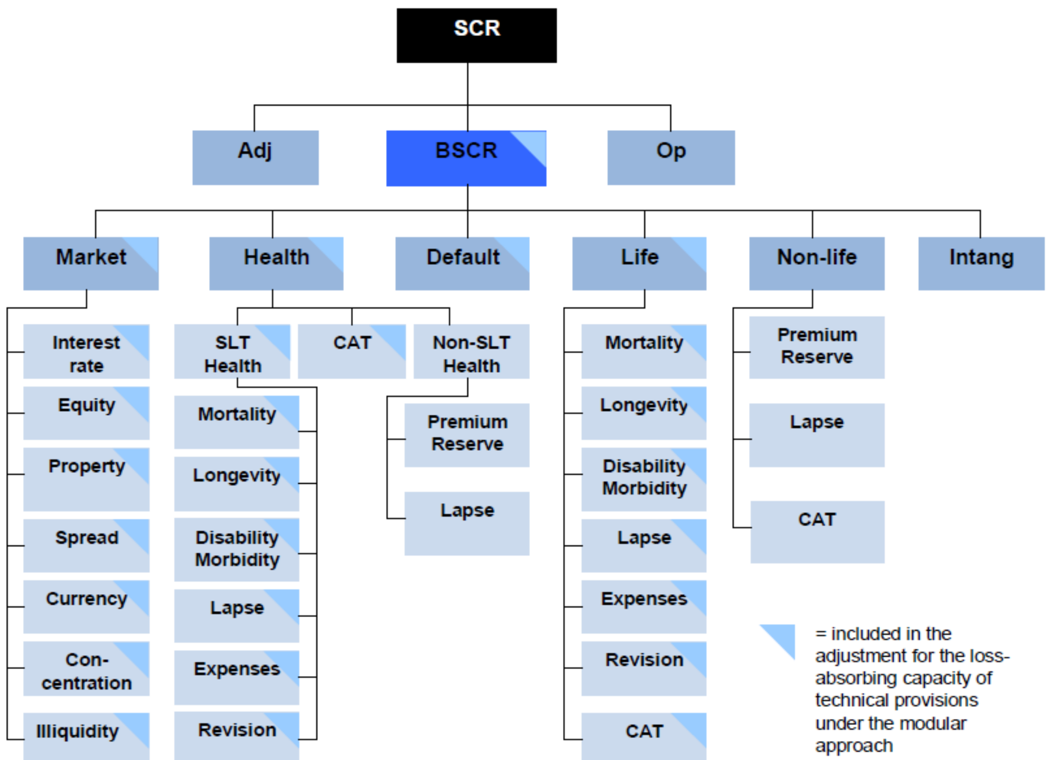
Solvency 2 takes into account the diversification effects. In the standard model, the "square-root formula" is used in order to calculate the aggregate solvency capital requirements (SCR_{ag}) for the BSCR and each risk module market, life, health and non-life (SCR_i)

$$SCR_{ag} = \sqrt{\sum_{i,j} CorrSCR_{i,j} \cdot SCR_i \cdot SCR_j} \tag{1.8.1}$$

where $CorrSCR_{i,j}$ stands for the correlation between the modules i and j.

Although it is an actual problem for the firms, there is currently no specification about the allocation of the capital due to the diversification effect within Solvency 2. Nevertheless, there are some examples on this subject in some documents provided by EIOPA. The standard formula for the aggregation is used. For the allocation of the benefits due to diversification, the proportional method is used. It is specified that it has its drawbacks. It seems it has been used in order to simplify the computations. This paper will show other methods and will try to give an answer to the problem of the allocation of the benefits due to the diversification effects.

Figure 1.8.1: SCR's structure



Chapter 2

The methods

2.1 Preliminary

Before describing the different methods and their possible, or not, application in case of the capital allocation problem, it is necessary to introduce some concepts and definitions.

Allocation principle

We will use the definition introduced by Denault [2]. An **allocation principle** is a function $\Pi : A \to \mathbb{R}^n$, where A is the set of risk capital allocation problems: pairs (N, ρ), with N a finite set of n portfolios and ρ a coherent risk measure, that maps each allocation problem (N, ρ) into a unique allocation:

$$\Pi : (N, \rho) \mapsto \begin{bmatrix} \Pi_1(N,\rho) \\ \Pi_2(N,\rho) \\ \cdot \\ \cdot \\ \cdot \\ \Pi_n(N,\rho) \end{bmatrix} = \begin{bmatrix} K_1 \\ K_2 \\ \cdot \\ \cdot \\ \cdot \\ K_n \end{bmatrix} \text{ such that } \sum_{i=1}^{n} K_i = \rho(X)$$

where K_i stands for the allocated capital of the i_{th} portfolio.

Coherent allocation principle

As above, we will use the definition given by Denault [2]: an allocation principle Π is coherent if it satisfies the following properties:

1. **No undercut**

   $$\sum_{i \in M} K_i \leq \rho(\sum_{i \in M} X_i) \textit{ for all } M \subseteq N,$$

   If a portfolio joins the firm, it cannot be allocated more risk capital than it can possible have brought to the firm. It could be compared with the sub-additivity of a risk measure.

2. **Symmetry** If by joining any subset $M \subseteq N \backslash \{i, j\}$ portfolios i and j both make the same contribution to the risk capital, then $K_i = K_j$. This means a portfolio allocation depends only on its contribution to the risk within the firm.

3. **Riskless allocation**

   $$K_n = \rho(\alpha r_f) = \alpha$$

   if the n^{th} portfolio is a riskless instrument.

All these properties have the goal to provide a *reasonable* risk allocation principle. They complete the fact that it is necessary to have, according to Denault [2], a *reasonable* risk measure, aka coherent.

Preliminaries on game theory

It is possible to define the game theory as the study of situations where players adopt various strategies to best attain their individual goals [2]. In particular, cooperative game theory has teams as the central unit. Given a set of agents (individuals), a cooperative game defines how well each coalition of can do for itself [11].

So, the goal of each player (equivalent to b.u. in our context) is to minimize the cost he incurs (the SCR in our context). To calculate the cost, it is necessary to introduce the concept of a *cost function*:

$$c \quad : \quad 2^N \quad \rightarrow \quad \mathbb{R} \ .$$

c associates a real number *c(S)* to each subset S of *N*, which is a finite set of n players.

We can see there are similarities between our problem and the cooperative game theory. Thus game theory provides an excellent framework on which to cast the allocation problem. There have been a lot of studies of the problem encountered here within the game theory area [2].

Strongly sub-additive game

A game is strongly sub-additive if its cost function c satisfies

$$c(S) + c(T) \geq c(S \cup T) + c(S \cap T)$$

for all coalitions $S \subseteq N$ and $T \subseteq N$.

2.2 The proportional method

This method is the simplest and the first one can think about. It aims to distribute the capital to the different modules according to their own SCR determined with a certain risk measure ρ. So, greater is the SCR, greater will be the allocated capital. Let us consider, for instance, two b.u. A_1 and A_2, with $SCR_1 = 40$ and $SCR_2 = 10$ respectively. Using the proportional method, A_1 will receive $^{40}/_{50} = 80\%$ of the benefits. Thus, A_2 will receive $^{10}/_{50} = 20\%$ of the benefits.

It can be easily implemented - as already noticed, it has been used in some papers of the EIPOA - but it does not take into account the marginal effects of the i^{th} module (aka the correlation with the other modules): this method considers that its risk is the same should the module be considered alone or within a coalition. Let us present the mathematical formula of the contribution of risk of the i_{th} b.u. associated to the risk measure ρ:

$$\rho^{prop}\left(X_i/X\right) = \frac{\rho\left(X_i\right)}{\sum_{j \in N} \rho\left(X_j\right)} \rho\left(X\right)$$

with $X = SCR(t)$ the SCR of the whole firm, $X_j = SCR_j(t)$ the SCR of the j_{th} b.u.

2.3 The marginal method

The motivation of the use of this method is to consider the marginal effects of the i^{th} module. In the proportional method, the main element that was considered is $\rho(X_i)$ for the i^{th} b.u. In this method, we take into account the difference between the risk of the coalition (aka the aggregated SCR) and the risk of the coalition without this module (aka the aggregated SCR of the coalition that does not include the considered b.u.). In this way, the marginal effect due to the i^{th} b.u. is the key element to determine how much benefit should be allocated to this module.

The example with the two b.u. A_1 and A_2 is quite trivial: If we consider that the aggregated SCR for the coalition of the two b.u. is 42, then the marginal effect due to A_1 is $42 - 10 = 32$ and to A_2 is $42 - 40 = 2$. In this case, $\frac{32}{32+2} = 94\%$ of the benefits will be allocated to A_1. Thus 6% to A_2.

The fact that only the effect of the i^{th} b.u. with the main coalition is taken into account is the main limitation of this method. Indeed, it does not consider any correlation between the different b.u. nor the effect on other coalitions, as only the whole is taken into account. The use of the Shapley value helps to limit this drawback.

Let us present the mathematical formula of the contribution of risk of each single segment i associated to risk measure ρ. To do so, it is necessary to introduce the mathematical definition of the marginal risk MR_i due to the i^{th} b.u. representing the marginal effect described before. Let us consider an insurance company with N b.u. and X_N its risk. If a fraction h of the i^{th} b.u. is removed, then the marginal risk on the insurance company due to the i^{th} b.u is:

$$MR_i(h,N) = \frac{\rho(X_N) - \rho(X_N - hX_i)}{h}$$

Thus MR_i represents how much does the risk measure vary when the contribution of the i_{th} b.u. varies. In case of discrete case (a b.u. is or is not in the coalition) h equals 1. Otherwise (only a part of a b.u. can join the insurance company) then h could be infinitely small.

From there, we can give the mathematical formula of contribution of risk using the marginal method (see for instance [6]):

$$\rho^{marg}\left(X_i/X\right) = \frac{MR_i\left(h,N\right)}{\sum_{j\in N} MR_j\left(h,N\right)}.$$

2.4 Shapley's method (Shapley value)

In game theory, Lloyd Shapley proposed a solution for a fair and unique distribution in a cooperative game with atomic players, i.e. in the discrete case. Let us consider a coalition of players that cooperates - meaning that the players want to find the best coalition to get the biggest profit. This method could be seen as a generalisation of the previous one. Indeed, the marginal effect that brings the i^{th} b.u. will be studied within all the possible coalitions. The problem is to determine the final distribution of generated surplus among the players. How much each of them should reasonably expect? The Shapley value gives one solution to this question.

Let N be a set of n players and c be a cost (or characteristic) function, that associates a real number *c(S)* to each subset S of *N*. In this case, *c(S)* describes the total expected sum of payoffs the members of S can obtain by cooperation. It is usually assumed that c is sub-additive, meaning $c(S \cup T) \leq c(S) + c(T)$, for all subsets S and T of N with empty intersection. We have to define the term of "dummy player": a player i is dummy if there is no diversification effect when he joins a coalition. So, he brings $c(i)$ by joining any coalition S.

To give a unique solution, the Shapley value has to satisfy the following properties:

1. **Symmetry:** If i and j are two equivalent players, i.e. they make the same contribution joining the subsets S that does not contain i nor j, then $\Pi(N,c)_i = \Pi(N,c)_j$.

2. **Efficiency:** For a dummy player we have: $\Pi(N,c)_i = c(i)$.

3. **Additivity:** Let us consider two games (N, c_1) and (N, c_2). Then we have:
   $\Pi(N, c_1+c_2) = \Pi(N,c_1)+\Pi(N,c_2)$, where the game (N, c_1+_2) is defined by $(c_1+c_2)(S) = (c_1(S))+(c_2(S))$ for all $S \leq N$.

Then it is possible to give an algebraic definition of the Shapley value for numerical computations. Let K^{Sh} be the Shapley value of the coalitional game *(N,c)*. Then we have

$$K_i^{Sh} = \Pi(N,c)_i = \sum_{S\in C_i} \frac{(s-1)!(n-s)!}{n!}(c(S) - c(S\backslash\{i\}))$$

where $i \in \mathbb{N}$, $s = card(S)$ and C_i represents the set of all coalitions of N that contain *i*.

In the case of the capital allocation problem, it is possible to associate the portfolios of a firm with the players of the game and the cost function c with the risk measure ρ. Unfortunately, using this method, Denault [2] concludes that a convincing proof of coherent allocation falls short.

Indeed, the risk measure ρ has to be positively homogeneous. This implies (for the proof: [2]) that ρ is linear, if c is strongly additive, a necessary property to insure the existence of the Shapley value. In this case, it is impossible to see a diversification effect. Indeed, the linearity means that we have the equality in the relation $\rho(X+Y) = \rho(X) + \rho(Y)$.

There are also other drawbacks with the Shapley value: the computation requires the evaluation of c for the $2^n - 1$ coalitions (if we consider that a coalition should not be empty) so the task could take too much time, if n is too large (from 3 [5]). Besides, the Shapley value deals only with the fact that the players are atomic. This is not reasonable from an capital allocation viewpoint, as only a part of a b.u. can join a coalition (see the next section).

2.5 The Aumann-Shapley method (Aumann-Shapley value)

We now consider the game theory for fractional players. It means that it is now possible to consider fractions of portfolios. The Aumann-Shapley value is the extended version of the Shapley value to non-atomic games. In this case, the discrete cost function c studied before has to be replaced by a real-valued cost function r that is less restrictive. To understand the utility of the Aumann-Shapley value, it is essential to introduce mathematical concepts:

Fuzzy value We will use the definition proposed by Denault [2]: a fuzzy value is a mapping assigning to each conditional game with fractional player (N, Λ, r) a unique per-unit allocation vector

$$\Phi : (N, \Lambda, r) \mapsto \begin{bmatrix} \Phi_1(N, \Lambda, r) \\ \Phi_2(N, \Lambda, r) \\ . \\ . \\ . \\ \Phi_n(N, \Lambda, r) \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ . \\ . \\ . \\ k_n \end{bmatrix},$$

with

$$\Lambda^t k = r(\Lambda).$$

In our case, the vector Λ represents the business volume of the b.u. We denote Λ_i the business volume of th i^{th} b.u. (aka its full involvement). The components of the vector k represent the per-unit allocation of risk capital to each portfolio, $k = (k_1, ..., k_n)$. r is a real-valued cost function. It is the extension of the cost function c seen before and it can be assimilated as the risk measure ρ in the case of capital allocation.

Coherent fuzzy value Again, we use the definition of Denault [2] for the coherent fuzzy value: Let r be a coherent risk measure. A fuzzy value

$$\Phi : (N, \Lambda, r) \mapsto k \in \mathbb{R}^n$$

is **coherent**, if it satisfies the properties defined below, and if k is an element of the **fuzzy core**:

- **Aggregation invariance:** *Suppose the risk measures* $r : \mathbb{R}^n \to \mathbb{R}$ and $s : \mathbb{R}^n \to \mathbb{R}$ satisfy $r(\lambda) = s(\Gamma\lambda)$ for some $m \times n$ matrix Γ and all λ such that $0 \leq \lambda \leq \Lambda$ componentwise. Then

  $$\Phi(N, \Lambda, r) = \Gamma^t \Phi(N, \Gamma\Lambda, s)$$

  This means that equivalent risks should receive equivalent allocations.

- **Continuity:** *The mapping Φ is continuous over the normed vector space M^n of continuously differentiable functions $r : \mathbb{R}_+^n \to \mathbb{R}$ such that $r(0) = 0$.*

  Continuity is wanted to ensure that similar risk measures yield similar allocations.

- **Non-negativity under r non-decreasing:** *If r is non-decreasing, in the sense that $r(\lambda) \leq r(\lambda^*)$ whenever $0 \leq \lambda \leq \lambda^* \leq \Lambda$ componentwise, then*

  $$\Phi(N, \Lambda, r) \geq 0.$$

  This means that "more risk" implies "more allocation".

- **Dummy player allocation:** *If i is a dummy player, in the sense that*

  $$r(\lambda) - r(\lambda^*) = (\lambda_i - \lambda_i^*) \frac{\rho(X_i)}{\Lambda_i},$$

  whenever $0 \leq \lambda \leq \Lambda$ and $\lambda^* = \lambda$ except in the i^{th} component, then

$$k_i = \frac{\rho(X_i)}{\Lambda_i}.$$

The dummy player, like the one defined for the Shapley value, is analogous to a riskless allocation.

- **Fuzzy core:** *The allocation* $\Phi(N, \Lambda, r)$ *belongs to the fuzzy core of a game* (N, Λ, r) *if for all* λ *such that* $0 \leq \lambda \leq \Lambda$,

$$\lambda^t \Phi(N, \Lambda, r) \leq r(\lambda)$$

as well as $\Lambda^t \Phi(N, \Lambda, r) = r(\Lambda)$.

Allocations obtained from a fuzzy core allow no undercut from any player and are considered fair.

The concept of a coherent fuzzy value is important, because it leads to a *fair* distribution. We remind that a fair allocation is in particular an allocation that verifies for a b.u., that the greater the risk the greater the allocation and that it is not possible for a b.u. to be allocated more risk capital than it can possibly have brought to the firm.

Non-atomic games: It is possible to give a formula of the Aumann-Shapley value. It is an extension given by Aumann and Shapley, and it is defined for the player i of N by the following formula:

$$\boxed{\phi_i^{AS}(N, \Lambda, r) = k_i^{AS} = \int_0^1 \frac{\partial r}{\partial \lambda_i}(\gamma \Lambda)\, d\gamma},$$

k_i^{AS} could be interpreted as a per-unit cost that is an average of the marginal cost of the i^{th} portfolio.

Lemma: If f is a k-homogeneous function, which means $f(\gamma x) = \gamma^k f(x)$, then: $\frac{\partial f(x)}{\partial x_i}$ is (k-1)-homogeneous. In our case r is 1-homogeneous. Indeed, the risk measures VaR and TVaR are 1-homogeneous (see proof hereinafter). Therefore, the previous formula is simplified and becomes:

$$\phi_i^{AS}(N, \Lambda, r) = k_i^{AS} = \frac{\partial r(\Lambda)}{\partial \lambda_i}.$$

We also have

$$K^{AS} = k^{AS} * \Lambda,$$

which represents the amount of risk capital allocated to each portfolio.

It is now possible to introduce the allocation coefficient $\mathcal{A}_i$ for the i^{th} b.u.:

$$\boxed{\mathcal{A}_i = \frac{(\frac{\partial r(\Lambda)}{\partial \lambda_i})\Lambda_i}{r(\Lambda)}}, \text{ for } i = 1, ..., n.$$

The Aumann-Shapley value as a coherent fuzzy value It is possible to prove (see [2]) the following theorem: If (N, r, Λ) is a game with fractional players, with a coherent cost function r that is differentiable at Λ, then the Aumann-Shapley value is a coherent fuzzy value.

It follows that the Aumann-Shapley value is the only linear coherent allocation principle, when the cost function is differentiable at Λ.

Nevertheless, in practice, the most used risk measures are VaR (non-coherent) and the TVaR. Both of them do not verify systematically all the previous axioms but it will be seen later that it is possible, under certain circumstances, to use them anyway. Besides, it is possible to obtain similar results with another method introduced hereinafter: the Euler principle.

2.6 Euler principle

We can use the Euler allocation principle for any 1-homogeneous and differentiable risk measure. Let denote X the portfolio-scale profit/loss which means: $X = \sum_{i=1}^n X_i$.

We call economic capital the capital that will be used in order to absorb any loss caused by the portfolio, it is denoted EC and we have $EC = \rho(X)$. Let $u = (u_1, u_2, ..., u_n)$ be the vector of weight factor which refers to some dynamics in the model, so: $X = X(u_1, ..., u_2) = \sum_{i=1}^n u_i X_i$. We assume that there is some variation of u, and so we introduce the function:$f_{\rho,X}(\text{u}) = \rho(X(u))$. We suppose that the distribution of X is fixed, then we can simplify the previous formula and put: $f_\rho(\text{u}) = \rho(X(u))$.

2.6.1 Risk distribution

The main question we try to answer to in this section is:"How much does the asset contribute to EC=$\rho(X)$?"

Definition : Let μ_i=$E[X_i]$, with $X_i = SCR_i$ the risk of the i_{th} asset and $E[X_i]$ its expected value.

- The total portfolio Return on Risk Adjusted Capital (RORAC) is defined by:

$$RORAC(X) = \frac{E[X]}{\rho(X)} = \frac{\sum_{i=1}^{m} \mu_i}{\rho\,(X)}.$$

- The portfolio related to RORAC of the i-th asset is defined by

$$RORAC(X_i/X) = \frac{E[X]}{\rho\,(X_i/X)} = \frac{\mu_i}{\rho\,(X_i/X)},$$

  where $\rho\,(X_i/X)$ is the risk contribution of X_i to $\rho(X)$

- Risk contributions $\rho\,(X_i/X)$, $i \in [|1;n|]$, to the portfolio $\rho(X)$ satisfy the full allocation property if:

$$\textstyle\sum_{i=1}^{n} \rho\,(X_i/X) = \rho(X).$$

- Risk contributions $\rho\,(X_i/X)$ are RORAC compatible, if there are some $\epsilon_i > 0$, such that

$$RORAC\,(X_i/X) > RORAC(X) \text{ implies } RORAC(X + hX_i) > RORAC(X) \text{ for all } 0 < h < \epsilon_i.$$

  Proposition Let ρ be a risk measure and f_ρ be the function that corresponds to ρ. Assume that f_ρ is continuously differentiable. If there are risk contributions

$$\rho\,(X_i/X),\ i \in [|1;n|],$$

  that are RORAC compatible for arbitrary expected values $\mu_1, ..., \mu_n$ of $X_1, ..., X_n$, then $\rho\,(X_i/X)$ is uniquely determined as:

$$\rho_{Euler}\,(X_i/X) = \frac{\partial \rho}{\partial h}(x + bx_j)\,|_{h=0} = \frac{\partial f_\rho}{\partial U_i}\,(1, \ldots, 1).$$

2.6.2 Risk measure formulas

In general, the formula of Euler's method with the value at risk as risk measure is very difficult to estimate. So, in the gaussian case we usually use a linear approximation. Risk contribution is then approximated by the following mathematical formula:

$$Var^{Euler} = \mu_i + \tfrac{cov(X,X_i)}{\sigma_X^2}\,(VaR_\alpha\,(X) + \mu_X)$$

where μ_i represents the mean of X_i and μ_X is the mean of X.

Chapter 3

Choice of the method

3.1 Choice of the risk measure

There is a large number of different risk measures, but we decided on purpose to present only two of them: VaR (Value-at-Risk) and the TVaR (Tail Value-at-Risk). Indeed, in practice, their use is common in a lot of firms and is required by regulations: Solvency II obliges the use of VaR while the TVaR is preferred by the Swiss Solvency Test. In theory, neither of them is perfect. VaR is not sub-additive, therefore non-coherent. The TVaR explains how much will cost on average the worst cases but its computation could not be easy. Both are not consistently derivative as wished.

Nevertheless, in practical situations, the performance of VaR and the TVaR could be different of what it would have been expected. For instance, one argument highlighting the TVaR is its coherence. But VaR could be, in some cases, more sub-additive than the TVaR. For example, VaR is sub-additive for elliptic distributions. Besides, it is sub-additive in most cases in the tail region, typically for $\alpha = 99\%$. This implies that VaR could be coherent in applications (see [9] or [5]).

That is why we choose to use VaR. As a matter of fact, we did not have actually the freedom to decide as BNP Paribas Cardif will have to follow the regulations required by Solvency II. Thus, we have to use VaR. Nonetheless, as explained earlier, this risk measure does not seem to be relevant at first sight but it is in fact usable for this purpose.

3.2 Choice of the allocation method

We have a choice between several allocation methods of capital. We have introduced and explained some of them previously, such as:

- the proportional method
- the marginal method
- the Shapley method
- the Aumann-Shapley method
- the Euler method

It is quite obvious that some of them cannot be used to calculate a fair allocation of the benefit due to the diversification. For the other, the choice is less trivial. Our conclusions are:

Both proportional and marginal methods are based on a very simplified principle of capital allocation that does not consider various elements and factors that can influence and change the contribution to the risk associated with each segment of the construction. So, according to these arguments, we consider that these methods were not relevant for this purpose.

The Shapley method is a discrete method inspired by the theory of cooperative games. However, it requires the computation of the cost function for 2^n possible coalitions, which means a huge computation time. It will eventually make the program running very slowly, which is not possible in our case because the speed of our program matters. Besides, in practice, it is not well adapted, as the b.u. could consider not to enter fully in a coalition. For instance, it could choose to join the coalition at 60 % only. So, the Shapley method is not relevant in our case.

After removing the 3 previous methods, there are only the Aumann-Shapley method and the Euler method remaining. It turns out that they are equivalent, because in our case we are using VaR, which is 1-homogeneous. Indeed, we saw that, using the Aumann Shapley value with a 1-homogeneous risk measure, we have:

$$k_i^{AS} = \frac{\partial r(\Lambda)}{\partial \lambda_i}$$

If we consider $\Lambda = (1, ..., 1)$ (Let us recall that this vector represents the full involvement of each b.u.), $\lambda_i = u_i$ and $r = f_\rho$, then

$$k_i^{AS} = \frac{\partial f_\rho}{\partial U_i}(1, \ldots, 1) = \rho_{Euler}(X_i/X).$$

This explains the fact that there is *a priori* no difference in using one of this method. It will be seen later that this is not entirely true when implementing the algorithm.

There are other methods proposed in the literature. We decided here to discuss only the main ones and we did not want to drown the reader in the middle of countless of methods.

Chapter 4

The algorithms

4.1 Significance of the results of the algorithm

4.1.1 Case of allocation with Aumann-Shapley method

4.1.1.1 Without correlation

For a better understanding, we start with a simple case of two independent random variables representing the losses. The goal is to simulate a set of losses following a distribution specified for each portfolio. The random variables need two main parameters: σ is the volatility of the portfolio, μ is its average yield.

Then we take VaR of level 99.5%, which gives the Solvency Capital Requirement for each portfolio (recorded SCRi for the i^{th} portfolio). That means the SCR enables the b.u. to manage 99.5% of cases.

As the portfolios are assumed to be independent (Their covariance is null, and we can anticipate a diversification profit), we simply add up the both vectors of simulations, and VaR of level 99.5% provides us the SCR for the entire firm (recorded SCRG)

Then we build the diversification profit (here as positive) from the difference between the sum of SCRi (before risk aggregation) and the SCRG.

Before continuing, we must check if the benefit is in fact positive. Now we can compute the allocations of profit for each portfolio (here we build positive ones). The Aumann-Shapley method guarantees that the allocation for the i^{th} portfolio is the derivative of VaR relative to λ (λ is the involvement of each portfolio). We make the choice to compute the derivative manually, namely (f(X(1+h)+Y)-f(X+Y))/h, with h very small (0.0001 for instance) and $f = VaR$ here. This latter is the allocation for X.

For more portfolios, we simply add a "for loop" over i between 1 and N, where N is the number of portfolios.

Let us observe that simulating losses or values of portfolios is the same: Value of a portfolio X equals the initial value X_0 minus losses described by a random variable L: $X = X_0 - L$. Then we see that simulating the losses and taking VaR of level 99.5%, is the same as simulating a portfolio with the opposite distribution (because $X = X_0 - L$, or the same distribution if the density respects parity), adjusting the average to X_0 (actually adding X_0 to the average if initially different of 0) and taking VaR of level 0.05% (because $1 - 0.995 = 0.005$).

4.1.1.2 Including correlation

Empirically, we can discern three main cases of correlation:

The first one that is mentioned above, the case of non-correlation, in which the random variables are non correlated, so that their covariance is null for two random variables. For many random variables, we use a correlation matrix, so the diagonal is full of 1 (because the diagonal represents the correlation of each variable with itself)and the rest of the matrix is 0. In this case, given the total independence of the random variables, there will be a diversification profit. For instance we can think to a portfolio containing only stocks of a data processing start-up, nd a second portfolio containing only options on brent. Even if actually, all markets are correlated, we can imagine the independence of these two portfolios.

The second case is that of total correlation: the covariance is 1 for two random variables, and for more, the correlation matrix is full of 1. In this case, the diversification profit is null. The simplest case is the following: two portfolios containing the same stocks in proportion. Another example of strength correlation may be a portfolio containing stocks of oil companies and a second containing options over brent.

The last case is that of totally inverse correlation : In this case, the diversification profit is maximum. Actually the losses of one are the gains of the other. We may think to a portfolio composed of stocks and a second composed by "put options" over these stocks. A Put is a derivative in finance which inversely grows when the associated stock falls.

You will therefore interpret the results for the diversification profit including that.

4.1.2 Case of allocation with Euler method

The goal remains the same: showing existence of a profit diversification, then allocate it between different b.u.

In this algorithm, we first simulate a correlation matrix. Then we decompose it by the Cholesky algorithm. We simulate the different portfolios in a matrix, and we make the product of the Cholesky decomposition with. After that, we attribute a weight to each portfolio. We compute each b.u. SCR (recorded SCR_i), then for the b.u. union, and the difference between the firsts and the second gives us the profit.

Conclusion

The goal of this project has been to explore different methods to allocate the diversification profit. We could see that existing methods are not trivial, except natural methods like the proportional, and creating a new method is very difficult due to these two points. Hence, we have chosen the best existing methods (from our point of view), to implement it. The subject of diversification profit and allocation is interesting, because it enables to better understand or discover different financial, accounting and mathematical concepts. The implementation phase has enabled us to face real problems, and apply the theory to them.

This project kept us in suspense during the whole scholar year, and has been for us a good training for our future Master thesis. We have in particular seen that is important to verify mathematical hypothesis to apply theorems, otherwise we could obtain aberrations, however in practice, hypothesis could not always be verified, although we have to use the theorem anyway. Related with the precaution with which obtained results have to be considered, and they have to be checked, e.g. by simulations.

Appendix A

Euler Method R Code

```
##EULER Method
## Correlated Portfolios Case
##We choose to work with Gaussian vectors to ensure subbativity of VaR


###############***************CODE***************#######################


#covariance matrix building, size p*p=n:
n=25
v=50*runif(n,0,1)
v
p=sqrt(n)
M=matrix(v,nrow=p)

M # covariance matrix

for(i in 1:p){
  for(j in 1:p){
M[j,i]=M[i,j]
M[i,i]=M[i,i]+100
}
}
M # matrix becomes positive defined

# we use Cholesky decomposition

Cholesky=chol(M)
Cholesky

# Global portfolio building X_u:

# s times simulation  of Gaussian distribution:

s = 1000000 # number of simulations
G = rnorm(p*s, 0, 1)
G = matrix(G,nrow =p)

X=t(Cholesky)%*%G
```

```
# weigth vector:

u = matrix(rep(1/sqrt(3), p))
X_u = t(u)%*%X  # global portfolio



#Computation of each initial SCR i (for each b.u.):

VaR=matrix(NA,ncol=p)# risk measure used

for(i in 1:p){
  VaR[i]=quantile(X[i,],0.995)
}

VaR

Scrg=sum(VaR)
Scrg


## SCR computation for b.u. union (then including aggregation):


VaRG=quantile(colSums(X),0.995)
VaRG## value is lower than sum(VaR). That shows that VaR is sub-additive in Gaussian case.


### Contribution of real risk contribution of each b.u. i with Euler method:

VaR_Euler = matrix(NA,ncol=p)

for(i in 1:p){
  #Risk contribution to b.u. i:
  VaR_Euler[i]=(cov(as.vector(X[i,]),as.vector(X_u), method="pearson")*(quantile(X_u,0.995)-
  mean(X_u)))/(sd(as.vector(X_u))^2)
}
VaR_Euler

Somme=sum(VaR_Euler)# sum of p contributions
Somme # equals VaRG

### Profit computation:

B=Scrg-VaRG
B

## Profit allocation:
Allocation=matrix(NA,ncol=p)
for(i in 1:p){
  Allocation[i]=VaR[i]-VaR_Euler[,i]
}
Allocation
sum(Allocation)## equals profit B
```

Appendix B

Computation with Data from BNP using Euler Method

```

These are the different  correlation matrix that we received from BNP Cardif:
    SCR_GlOBAL_COR=read.csv("C:\\Users\\Alexandre\\Documents\\BE\\matrices de cor\\
    SCR_GLOBAL_COR.csv",sep=’;’,head=FALSE)
    SCR_MARKET_UP_COR=read.csv("C:\\Users\\Alexandre\\Documents\\BE\\matrices de cor\\
    SCR_MARKET_UP_COR.csv",sep=’;’,head=FALSE)
    HEALTH_GLOBAL_COR=read.csv("C:\\Users\\Alexandre\\Documents\\BE\\matrices de cor\\
    HEALTH_GLOBAL_COR.csv",sep=’;’,head=FALSE)
    HEALTH_NONSLT_COR=read.csv("C:\\Users\\Alexandre\\Documents\\BE\\matrices de cor\\
    HEALTH_NONSLT_COR.csv",sep=’;’,head=FALSE)
    HEALTH_SLT_COR=read.csv("C:\\Users\\Alexandre\\Documents\\BE\\matrices de cor\\
    HEALTH_SLT_COR.csv",sep=’;’,head=FALSE)
    SCR_LIFE_COR=read.csv("C:\\Users\\Alexandre\\Documents\\BE\\matrices de cor\\
    SCR_LIFE_COR.csv",sep=’;’,head=FALSE)
    SCR_NON_LIFE_COR=read.csv("C:\\Users\\Alexandre\\Documents\\BE\\matrices de cor\\
    SCR_NON_LIFE_COR.csv",sep=’;’,head=FALSE)
    COR=read.csv("C:\\Users\\Alexandre\\Documents\\BE\\matrices de cor\\
    COR.csv",sep=’;’,head=FALSE)
    SCR_SCR=read.csv("C:\\Users\\Alexandre\\Documents\\BE\\matrices de cor\\
    SCR_SCR.csv",sep=’;’,head=FALSE)
    SCR=read.csv("C:\\Users\\Alexandre\\Documents\\BE\\matrices de cor\\
    SCR.csv",sep=’;’,head=TRUE)
    SCR
    truc=quantile(rnorm(100000000),0.995)
    truc
    # 2,576
    VAR=function(M){
    +   p=dim(M)[1]
    +   M=as.matrix(M)
    +   while((min(eigen(M)$values>0)==0)){M=t(M)%*%M}
    +
    +   Cholesky=chol(M)
    +   Cholesky
    +   #Construction of the global portfolio:
    +   #Simulation :
    +   s = 1000000 # number  of simulations
    +   G = rnorm(p*s, 0, 1)
    +   G = matrix(G,nrow =p)
    +   X=t(Cholesky)%*%G
```

```
+   #weight vector:
+   u = matrix(rep(1/sqrt(3), p))
+   X_u = t(u)%*%X  # global portfolio
+   # SCR  for each segment i:
+   VaR=matrix(NA,ncol=p)# the risk measure used
+   for(i in 1:p){
+     VaR[i]=quantile(X[i,],0.995)
+   }
+   VaR
+   Scrg=sum(VaR)
+   Scrg
+   ## SCR for b.u. i union:
+   VaRG=quantile(colSums(X),0.995)
+   VaRG## this value is smaller than Sum(Var) which means that in this case the Var
 is sub addtive
+   ###the real contrubition for each segment i, using the Euler's method:
+   VaR_Euler = matrix(NA,ncol=p)
+   for(i in 1:p){
+     # the real contrubition for each segment i
VaR_Euler[i]=(cov(as.vector(X[i,]),as.vector(X_u), method="pearson")*
(quantile(X_u,0.995)-mean(X_u)))/(sd(as.vector(X_u))^2)
+   }
+   VaR_Euler
+   Somme=sum(VaR_Euler)# Sum of  p contributions
+   Somme # it is equal to au VaRG
+   ### bénéfice:
+   B=Scrg-VaRG
+   B
+   ## Allocation of the benefice B:
+   Allocation=matrix(NA,ncol=p)
+   for(i in 1:p){
+     Allocation[i]=VaR[i]-VaR_Euler[,i]
+   }
+   Allocation
+   sum(Allocation)##  it is equal to the benefice
+   return(VaR_Euler)
+ }

> Entité1=as.vector(SCR[,3])

SCR contains the integrity of the input array
> # we will change the correlation matrix SCR MARKET_UP_COR in covariance matrix
> ## Market risk
> market=c(Entité1[4],Entité1[7],Entité1[10],Entité1[11],Entité1[12])
> sigma=market/2.576# property of the value at risk
> sigma
> COV_MARKET=SCR_MARKET_UP_COR[-(3:4),-(3:4)]*(sigma%*%t(sigma))
COV_ MARKET  is the  new covariance matrix
> VAR(COV_MARKET)
> ## Risque LIFE
> life=Entité1[c(18,19,20,24,25,26,27)]
> life
> sigmalife=life/2.576
> COV_LIFE=SCR_LIFE_COR[-c(2,3,6),-c(2,3,6)]*(sigmalife%*%t(sigmalife))
> life
> ## Risque HEALTH SLT
```

```
> slt=Entité1[c(31,32,36,40,41,42)]
> slt
> sigmaslt=slt/2.576
> COV_HEALTH_SLT=HEALTH_SLT_COR[-c(1,6),-c(1,6)]*(sigmaslt%*%t(sigmaslt))
> slt
> ## Risque HEALTH
> health=Entité1[c(30,43,46)]
> sigmahealth=health/2.576
> COV_HEALTH=HEALTH_GLOBAL_COR*(sigmahealth%*%t(sigmahealth)
> health
> ## Risque non life
> nonlife=Entité1[c(49,50,51)]
> sigmanonlife=nonlife/2.576
> COV_NONLIFE=SCR_NON_LIFE_COR*(sigmanonlife%*%t(sigmanonlife))
> nonlife
> ## RISk entité 1
> global=Entité1[c(1,14,17,29,48)]
    > sigmaglob=global/2.576
> COV_GLOB=SCR_GlOBAL_COR*(sigmaglob%*%t(sigmaglob))
```

Appendix C

Aumann-Shapley Method R Code

C.1 Simple case of two portfolios without correlation

```
#Aumann-Shapley Method
#VaR derivative according to lambda[i]
# Simple case of Gaussian distribution for losses for two portfolios supposed independent
######################
alpha=0.995 #level of VaR
sigma1=4;sigma2=4
mu1=0;mu2=0 #averaged yields
#  (so (-1)*Drift because negative #loss=gain).
X0=100;Y0=200;P0=X0+Y0  #initial of each  portfolio at t=0



n=100000 #number of simulations


X=rnorm(n,mu1,sigma1) #Potential losses #for portfolio X
Y=rnorm(n,mu2,sigma2) #losses for Y

SCR1=quantile(X,alpha) # worst loss for X then for Y
SCR2=quantile(Y,alpha)

#X and Y are supposed independent, so we sum vectors
# of N simulated losses, then we take
#the worst value of 99.5 % of cases
SCRG=quantile(X+Y,alpha)
SCR1;SCR2;SCRG

#"Diversification profit"


SommeSCR=SCR1+SCR2 #we sum SCR of each
#b.u. "before diversification"
SommeSCR

B=SommeSCR-SCRG  #DIVERSIFICATION PROFIT (we want it >0)
B

#ALLOCATION of DIVERSIFICATION PROFIT
```

```
#Derivatives of VaR (Aumann-Shapley method)
h=0.00001 #derivating step
all1=(1/h)*(quantile((X0-X)*(1+X0/P0*h)+(Y0-Y)*(1-X0*X0*h/(Y0*P0)),1-alpha)-
quantile(X0-X+Y0-Y,1-alpha))
#original formula: allocation computing is about portfolios then:
#(1/h)*(P0-quantile(X+Y,0.95)-
(P0-quantile(X*(1+h)+Y,0.95)), P0
# simplify by themselves.
all2=(1/h)*(quantile((Y0-Y)*(1+Y0/P0*h)+(X0-X)*(1-Y0*Y0*h/(X0*P0),1-alpha)-
quantile(X0-X+Y0-Y,1-alpha))
#the sum of weights must always equal 1

all1;all2

all1+all2 #this must equal the profit B
B-(all1+all2) # must be near 0
```

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